

EULERIAN PARTITIONS FOR CONFIGURATIONS OF SKEW LINES

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ABSTRACT. In this paper, which is a complement of [1], we study a few elementary invariants for configurations of skew lines, as introduced and analyzed first by Viro and his collaborators. We slightly simplify the exposition of some known invariants and use them to define a natural partition of the lines in a skew configuration.

We also describe an algorithm which constructs a spindle-permutation for a given switching class, or proves non-existence of such a spindle-permutation.

1. INTRODUCTION

A *configuration of n skew lines in \mathbb{R}^3* or a *skew configuration* or an *interlacing of skew lines* is a set of n non-intersecting lines in \mathbb{R}^3 containing no pair of parallel lines.

Skew configurations are only considered up to *rigid isotopies* (continuous deformations of skew configurations or, equivalently, isotopies of the ambient space under which lines remain pairwise skew lines).

The study and classification of configurations of skew lines up to isotopy was initiated by Viro [12] and pursued by Viro, Mazurovskii, Borobia-Mazurovskii, Drobotukhina and Khashin, see for example [2, 3, 5, 6, 8, 9, 13]. The survey paper [13] (and its updates available on the authors web-sites and on the arXiv) contains historical information and is a good introduction into the subject and its higher-dimensional generalizations. Most of these results are also exposed in the survey paper [4] from which we borrow some terminology not used in the original work of Viro's school.

A *spindle* (or *isotopy join* or *horizontal configuration*) is a particularly nice configuration of skew lines in which all lines intersect an oriented additional line A , called the *axis* of the spindle. Its isotopy class

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²Partially supported by the Chateaubriand postdoctoral fellowship funded by the French government.

Key words and phrases. Configurations of skew lines, linking matrix, Switching graph, Eulerian graph, Spindles, permutation.

is completely described by a *spindle-permutation* $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ encoding the order in which an open half-plane revolving around its boundary A intersects the lines during a half-turn (see Section 5 for the precise definition). A *spindle-configuration* is a skew configuration isotopic to a spindle.

Three types of combinatorial moves (described in Section 5) of a spindle permutation yield isotopic spindles-configurations and generate an equivalence relation, called the *spindle-equivalence* relation, on permutations of $\{1, \dots, n\}$.

Isotopy classes of spindle-configurations are well understood by the following combinatorial description, given in [1]:

Theorem 1.1. *Two spindle-permutations σ, σ' give rise to isotopic spindle-configurations if and only if σ and σ' are spindle-equivalent.*

Orienting and labeling all lines of a skew configuration, one gets a *linking matrix* encoding isotopy classes for pairs of oriented skew lines. The associated *switching class* or *homology equivalence class* is independent of labels and orientations. A result of Khashin and Mazurovskii [7, Theorem 3.2] states that homology-equivalent spindles (spindles defining the same switching class) are isotopic. Thus we have:

Corollary 1.2. *Two spindle-permutations define the same switching class if and only if they are spindle-equivalent.*

In this paper we define the Euler partition for a switching class. The definition depends on the parity of the order and can be refined to an Euler tree for switching classes of even order.

The last part of the paper describes an algorithm for computing a spindle-permutation (or proving its non-existence) for a given switching class.

The sequel of this paper is organized as follows: Sections 2 and 3 are devoted to (various aspects of) switching classes. Section 4 deals with the Eulerian partition induced by the switching classes. Section 5 introduces spindle-structures. In Section 6 we describe an algorithm which computes (or proves non-existence of) a spindle-permutation (which is unique up to spindle-equivalence by Corollary 1.2) having a linking matrix of given switching class.

2. LINKING MATRICES AND SWITCHING CLASSES

Pairs of oriented under- or over-crossing curves (as arising for instance from oriented knots and links) can be encoded by signs ± 1 as shown in Figure 1.

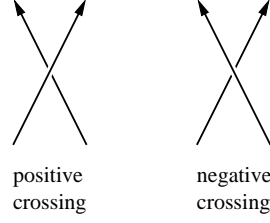


FIGURE 1. Positive and negative crossings

The *sign* or *linking number* $\text{lk}(L_A, L_B) \in \{\pm 1\}$ between two oriented skew lines $L_A, L_B \subset \mathbb{R}^3$ was introduced by Viro [12]. The *linking matrix* of a configuration involving n oriented and labeled skew lines L_1, \dots, L_n is the symmetric $n \times n$ matrix X with diagonal coefficients $x_{i,i} = 0$ and $x_{i,j} = \text{lk}(L_i, L_j) \in \{\pm 1\}$ for $i \neq j$.

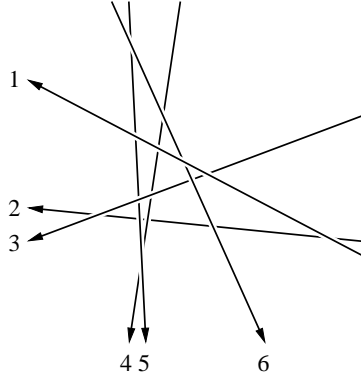


FIGURE 2. A configuration of 6 labeled and oriented skew lines

Figure 2 shows a labeled and oriented configuration of six skew lines with linking matrix

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

Two symmetric matrices X and Y are *switching-equivalent* if

$$Y = D P^t X P D$$

where P is a permutation matrix and D is a diagonal matrix with $d_{i,i} \in \{\pm 1\}$. Since P and D are orthogonal, we have $(PD)^{-1} = D^t P^t = DP^t$.

Switching-equivalent matrices are thus conjugate and have the same characteristic polynomial.

Proposition 2.1. *All linking matrices of a fixed configuration of skew lines are switching-equivalent.*

Proof. Relabeling the lines conjugates a linking matrix X by a permutation matrix. Inverting the orientation of some lines amounts to conjugation by a diagonal ± 1 matrix. \square

Remark 2.2. *The terminology “switching classes” (many authors speak of “two-graphs” which is the standard terminology for the underlying combinatorial object) is motivated by the following combinatorial interpretation and definition of switching classes.*

Two finite simple (loopless and no multiple edges) graphs $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ are **switching-related** with respect to a subset of vertices $V_- \subset V$ if their edge-sets E_1, E_2 coincide on $(V_- \times V_-) \cup ((V \setminus V_-) \times (V \setminus V_-))$ and are complementary on $(V_- \times (V \setminus V_-)) \cup ((V \setminus V_-) \times V_-)$. A switching class of graphs is an equivalence class of switching-related graphs, see Figure 3 for two graphs in a common switching class.

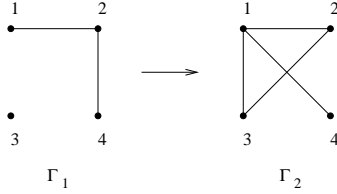


FIGURE 3. Γ_1 and Γ_2 are switching-related with respect to $\{1, 2\} \subset \{1, 2, 3, 4\}$

Encoding adjacency, respectively non-adjacency, of distinct vertices by 1, respectively -1 , yields a bijection between switching classes of graphs and switching classes of matrices. Conjugation by permutation matrices corresponds to relabeling the vertices of a graph Γ . Conjugation by a diagonal ± 1 -matrix corresponds to the substitution of Γ by a switching-related graph.

3. SWITCHING CLASSES AND VORTICITY

The set of vorticities (also called *homological equivalence class* or *chiral signature*), introduced by Viro [12], is a classical and well-known invariant for configurations of skew lines. We sketch below briefly the well-known proof that it corresponds to the switching class of an associated linking matrix.

We prefer to work with switching classes corresponding to symmetric matrices (up to conjugation by signed permutation matrices) with zero diagonal and off-diagonal coefficients in $\{\pm 1\}$.

The *vorticity* $\text{vort}(L_i, L_j, L_k)$ of three lines (see [12, Section 2] or [4, Section 3]) is defined as the product $x_{i,j}x_{j,k}x_{k,i} \in \{\pm 1\}$ of the signs for the corresponding three crossings. The result is independent of the chosen orientations for L_i, L_j, L_k , classifies the skew-configuration $\{L_i, L_j, L_k\}$ up to isotopy and yields an invariant

$$\{\text{triplets of lines in configurations of skew lines}\} \longrightarrow \{\pm 1\}.$$

Let us remark that almost all authors use the terminology *linking coefficient* instead of vorticity. This is slightly confusing since the linking coefficient denotes also the isotopy type of a pair of oriented skew lines.

The *set of vorticities* is the list of vorticities $\text{vort}(L_i, L_j, L_k)$ for all triplets of lines $\{L_i, L_j, L_k\}$ in a configuration of skew lines.

Sets of vorticities (defining a *two-graph*, see [14]) and switching classes are equivalent. Indeed, vorticities of a configuration of skew lines \mathcal{C} can easily be retrieved from a linking matrix for \mathcal{C} . Conversely, given all vorticities $\text{vort}(L_i, L_j, L_k)$ of a configuration $\mathcal{C} = \{L_1, \dots, L_n\}$, choose an orientation of the first line L_1 . Orient the remaining lines L_2, \dots, L_n such that they cross L_1 positively. A linking matrix X for \mathcal{C} is given by $x_{1,i} = x_{i,1} = 1$ for i such that $2 \leq i \leq n$ and $x_{a,b} = \text{vort}(L_1, L_a, L_b)$ for $2 \leq a \neq b \leq n$.

Two configurations of skew lines are *homologically equivalent* (the terminology refers to properties of the complement, endowed with a suitable extra-structure, of a configuration in \mathbb{RP}^3) if there exists a bijection between their lines, which preserves all vorticities. Two configurations are *homologically equivalent* if and only if they have switching-equivalent linking matrices.

The sign indeterminacy in linking matrices representing switching classes makes their use more difficult. A satisfactory answer addressing this problem will be given in Section 4.1 for switching classes of odd order. For even orders, there seems to be no completely satisfactory way to get rid of all sign-indeterminacies, see Section 4.2.

4. EULER PARTITIONS

In this section we study invariants of computational cost $O(n^2)$ for switching classes of order n .

The behaviour of switching classes depends on the parity of their order.

Switching classes of odd order $2n - 1$ are in bijection with Eulerian graphs. This endows the lines of a skew configuration consisting of an odd number of lines with a *semi-orientation* (a canonical orientation, up to global change) which we call the *Eulerian semi-orientation*. An Eulerian semi-orientation induces a partition of the lines into equivalence classes by counting their number of positive crossings. We consider the case of odd order in Section 4.1.

The situation for switching classes of even order $2n$ is more complicated. We replace Eulerian graphs appearing for odd orders by a suitable kind of rooted binary trees which we call *Euler trees*. The leaves of the Euler tree induce again a natural partition, called the *Euler partition*, of the set of lines into equivalence classes of even cardinalities. Section 4.2 deals with the even case.

4.1. Switching classes of odd order - Eulerian semi-orientations.

A simple finite graph Γ is *Eulerian* if all its vertices are of even degree. Figure 4 shows all seven Eulerian graphs on 5 vertices.

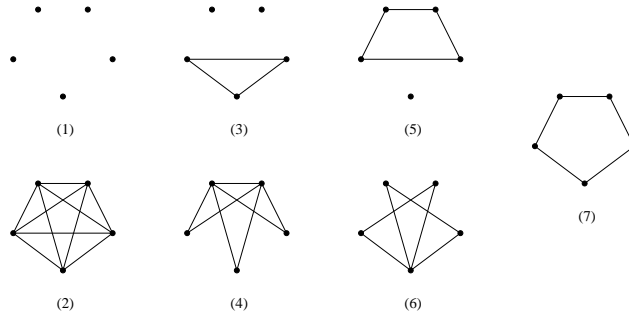


FIGURE 4. All Eulerian graphs on 5 vertices

The following well-known result goes back to Seidel [10].

Proposition 4.1. *Eulerian graphs with an odd number $2n - 1$ of vertices are in bijection with switching classes of order $2n - 1$.*

We recall a simple proof of Proposition 4.1 since it yields a fast algorithm for computing Eulerian semi-orientations on configurations with an odd number of skew lines.

Proof. Choose a representing matrix X of a switching class. For i such that $1 \leq i \leq 2n - 1$ define the number

$$v_i = \#\{j \mid x_{i,j} = 1\} = \sum_{j=1, j \neq i}^{2n-1} \frac{x_{i,j} + 1}{2}$$

counting all entries equal to 1 in the i -th row of X . Since X is symmetric, the vector (v_1, \dots, v_{2n-1}) has an even number of odd coefficients and conjugation of the matrix X with the diagonal matrix having diagonal entries $(-1)^{v_i}$ turns X into a matrix X_E having an even number of 1's in each row and column. The matrix X_E is well-defined up to conjugation by a permutation matrix and defines an Eulerian graph Γ_X with vertices $\{1, \dots, 2n-1\}$ and edges $\{i, j\}$ if $(X_E)_{i,j} = 1$. The Eulerian graph Γ_X is unique up to relabeling its vertices since switching with respect to a non-trivial subset of vertices destroys the Eulerian property of Γ_X . \square

A *semi-orientation* of a set of lines \mathcal{L} is an orientation of all lines in \mathcal{L} , up to global inversion of all orientations.

Let \mathcal{C} be a configuration of skew lines having an odd number of lines. Label and orient the lines of \mathcal{C} arbitrarily in order to get a linking matrix X . Inverting the orientations of all lines having an odd number of positive crossings we get a unique semi-orientation which we call the *Eulerian semi-orientation* of \mathcal{C} .

An *Eulerian linking matrix* X_E associated to an Eulerian semi-orientation of \mathcal{C} is uniquely defined up to conjugation by a permutation matrix. Its invariants coincide with those of the switching class of X_E but are slightly easier to compute since there is no sign ambiguity. In particular, some of them can be computed using only $O(n^2)$ operations.

An *Eulerian partition* of the set of lines of a configuration consisting of an odd number of lines is by definition the partition of the lines into subsets \mathcal{L}_k consisting of all lines involved in exactly $2k$ positive crossings for an Eulerian semi-orientation.

A few more invariants of Eulerian matrices are:

- (1) The total sum $\sum_{i,j} x_{i,j}$ of all entries in an Eulerian linking matrix X_E (this is of course equivalent to the computation of the number of entries equal to 1 in X_E). The computation of this invariant needs only $O(n^2)$ operations.
- (2) Its signature $\epsilon = \prod_{i < j} x_{i,j}$. The easy identity

$$\epsilon = (-1)^{(-n(n-1) + \sum_{i,j} x_{i,j})/4}$$

relates the signature to the total sum $\sum_{i,j} x_{i,j}$ of all entries in an Eulerian linking matrix.

- (3) The number of rows of X_E with given row-sum. These numbers yield of course the cardinalities of the sets $\mathcal{L}_0, \mathcal{L}_1, \dots$ and can be computed using $O(n^2)$ operations.

- (4) All invariants of the associated Eulerian graph (having edges corresponding to entries $x_{i,j} = 1$) defined by X_E , e.g. the number of triangles or of other fixed subgraphs. In particular, one can consider the number $a_{i,j}$ of edges joining a vertex of degree $2i$ to a vertex of degree $2j$.

For example, for 7 vertices, there are 54 different Eulerian graphs, 36 different sequences of vertex degrees (up to a permutation of the vertices), and 18 different numbers for the cardinality of 1's in X_E .

4.2. Switching classes of even order - Euler partitions. The situation in this case is more complicated and less satisfactory.

Given a matrix X representing a switching class with $2n$ vertices, there exists a natural partition of the $2n$ rows R of X into two subsets R_+ and R_- according to the sign

$$\epsilon_i = \prod_{j \neq i} x_{i,j} \prod_{s < t} x_{s,t}$$

associated to the i -th row of X . This sign is well-defined since switching (conjugation) with respect to a diagonal $\{\pm 1\}$ -matrix D multiplies both factors $\prod_{j \neq i} x_{i,j}$ and $\prod_{s < t} x_{s,t}$ by $\det(D) \in \{\pm 1\}$.

Since $\prod_i \epsilon_i = \prod_{i,j, i \neq j} x_{i,j} (\prod_{s < t} x_{s,t})^{2n} = 1$, both subsets R_+, R_- have even cardinalities.

If R_+ (or equivalently, R_-) is non-empty, it defines a symmetric submatrix X_+ of even size $\sharp(R_+)$ corresponding to all rows and columns with indices in R_+ . Iterating the above construction we get a partition $R_+ = R_{++} \cup R_{+-}$. This construction is most conveniently encoded by a rooted binary tree embedded in the oriented plane which we call the *Euler tree* of X : Draw a root R corresponding to the row-set R of X . If the partition $R = R_+ \cup R_-$ is non-trivial, join the root R to a left successor called R_- and a right successor called R_+ . The Euler tree of X is now constructed recursively by gluing the root R_{\pm} of the Euler tree associated to X_{\pm} onto the corresponding successor R_{\pm} of the root R .

The leaves of the Euler tree $T(X)$ of X correspond to subsets R_w (with $w \in \{\pm\}^*$) of even cardinalities $2n_w$ summing up to $2n$. Leaves of $T(X)$ define symmetric submatrices in X which we call *Eulerian*: All their row-sums are identical modulo 2 and can be chosen to be even, after a suitable conjugation. The row partition $R = R_+ \cup R_-$ of an Eulerian matrix X of even size is by definition trivial. The sign $\epsilon \in \{\pm 1\}$ defined as $\epsilon = \epsilon_i$ for i an arbitrary row of X is called the *signature* of the Eulerian matrix X . An Eulerian matrix of size 2

has always signature 1. For Eulerian matrices of size $2n \geq 4$ both signs can occur as signatures since changing the signs of the entries $x_{i,j}$, $1 \leq i \neq j \leq 3$, inverts the signature of an Eulerian matrix (and preserves the set of Eulerian matrices of even size ≥ 4). The signature of an Eulerian matrix encodes the parity of the number of edges in an Eulerian graph (having only vertices of even degrees) in the switching class of X .

The leaves of the Euler tree define a natural partition of the set of rows of X into subsets. We call this partition the *Euler partition*.

An Euler tree is *signed* if its leaves are endowed with signs ± 1 corresponding to the signs of the associated Eulerian matrices. An Euler tree is *weighted* if its leaves are endowed with strictly positive natural weights, a weight m corresponding to an Eulerian matrix of size $2m \times 2m$. A *signed weighted* Euler tree is both signed and weighted.

Example 4.2. *The symmetric matrix*

$$\begin{pmatrix} 0 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

yields the Euler partition

$$R_{--} = \{3, 5\} \cup R_{-+} = \{6, 10\} \cup R_{+} = \{1, 2, 4, 7, 8, 9\}$$

(where the Eulerian submatrix associated to R_{+} has signature 1). The associated signed Euler tree (with leaves of respective weights 1, 1 and 3) is depicted in Figure 5.

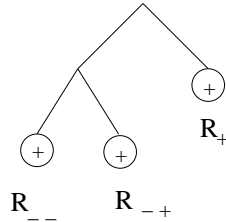


FIGURE 5. The signed Euler tree defined by Example 4.2

Example 4.3. *The characteristic polynomial of a linking matrix of a configuration of skew lines is in general weaker than its switching class: The linking matrices*

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

are in different switching classes: The row-partition $R = R_- \cup R_+$ of the first matrix is given by $R_- = \{2, 5, 7, 8\}$ and $R_+ = \{1, 3, 4, 6\}$ with associated Eulerian matrices X_- and X_+ both of signature 1. The second matrix is Eulerian with signature 1. On the other hand, they have the same characteristic polynomial

$$(t - 3)(t - 1)^2(t + 1)(t + 3)^2(t^2 - 2t - 11) .$$

This example is minimal in the sense that distinct switching classes of order less than 8 have distinct characteristic polynomials.

Let us mention a last invariant related to the Euler tree for a switching class X having even order $2n$. Let $R_1, \dots, R_m \subset R$ be the Euler partition of X . For $1 \leq i \leq j \leq m$, define numbers $a_{i,j} \in \{\pm 1\}$ by

$$a_{i,j} = \begin{cases} \prod_{t \neq s_{i_0} \in R_i} x_{s_{i_0}, t} \prod_{s, t \in R_i, s < t} x_{s, t} & i = j \\ \prod_{s \in R_i, t \in R_j} x_{s, t} & i \neq j \end{cases}$$

where $s_{i_0} \in R_i$ is a fixed element. Note that the number $a_{i,i}$ is the signature of the Eulerian matrix defined by the rows (and columns) of the set R_i . One can easily check that the numbers $a_{i,j}$ are well-defined.

Remark 4.4. *The equivalence relation induced on lines by the Euler partition is fairly coarse. It is for instance generally much rougher than the equivalence relation given by homologous lines defined by Viro [12].*

4.3. Enumerative aspects. It is natural to enumerate (signed) weighted Euler trees according to the total sum n of all weights.

The generating function $F(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ enumerating the number α_n of distinct weighted Euler trees with total weight n satisfies the equation

$$F(z) = \frac{1}{1-z} + (F(z) - 1)^2$$

(with $\alpha_0 = 1$ corresponding to the empty tree). Indeed, weighted Euler trees reduced to a leaf contribute $1/(1-z)$ to $F(z)$. All other weighted Euler trees are obtained by gluing two weighted Euler trees of strictly positive weights below a root and are enumerated by the factor $(F(z) - 1)^2$.

Solving for $F(z)$ we get the closed form

$$F(z) = \sum_{n=0}^{\infty} \alpha_n z^n = \frac{3(1-z) - \sqrt{(1-z)(1-5z)}}{2(1-z)}.$$

showing that

$$\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} = 5.$$

The first terms $\alpha_0, \alpha_1, \dots$ are given by

$$1, 1, 2, 5, 15, 51, 188, 731, 2950, 12235, \dots,$$

see also Sequence A7317 of [11].

Similarly, the generating function $F_s(z) = \sum_{n=0}^{\infty} \beta_n z^n$ enumerating the number β_n of signed weighted Euler trees (keeping also track of the signature of all leaves with weight ≥ 2) with total weight n satisfies the equation

$$F_s(z) = \frac{1+z^2}{1-z} + (F_s(z) - 1)^2.$$

We get thus

$$F_s(z) = \sum_{n=0}^{\infty} \beta_n z^n = \frac{3(1-z) - \sqrt{(1-z)(1-5z-4z^2)}}{2(1-z)}.$$

and

$$\lim_{n \rightarrow \infty} \frac{\beta_{n+1}}{\beta_n} = \frac{5 + \sqrt{41}}{2} \sim 5.7016.$$

The first terms β_0, β_1, \dots are given by

$$1, 1, 3, 8, 27, 104, 436, 1930, 8871, 41916, \dots,$$

see Sequence A110886 of [11].

5. SPINDLE STRUCTURES FOR SWITCHING CLASSES

A construction of Mazurovskii originally called the *isotopy join* (or *spindle*) associates a configuration of n skew lines to every permutation of n letters. We recall that a spindle is a configuration of skew lines with all lines intersecting an oriented auxiliary line A , called its *axis*. A *spindle-configuration* (or a *spindle structure*) is a configuration of skew lines isotopic to a spindle.

The orientation of the axis A induces a linear order $L_1 < \dots < L_n$ on the n lines of a spindle C . Each line $L_i \in C$ defines a plane Π_i containing L_i and the axis A .

A second oriented auxiliary line B (called a *directrix*) in general position with respect to A, Π_1, \dots, Π_n and crossing A negatively, intersects the planes Π_1, \dots, Π_n at points $\sigma(L_i) = B \cap \Pi_i$. One can assume $\sigma(L_i) \in L_i$ by a suitable rotation fixing $A \cap \Pi_i$ of the plane Π_i containing L_i . Since the orientation of B induces a linear order on the points $\sigma(L_i)$, we get a spindle-permutation (still denoted) $i \mapsto \sigma(i)$ of the set $\{1, \dots, n\}$ by identifying the two linearly ordered sets L_1, \dots, L_n and $\sigma(L_1), \dots, \sigma(L_n)$ in the obvious way with $\{1, \dots, n\}$. Figure 6 displays an example corresponding to $\sigma(1) = 1, \sigma(2) = 4, \sigma(3) = 2, \sigma(4) = 5, \sigma(5) = 3$.

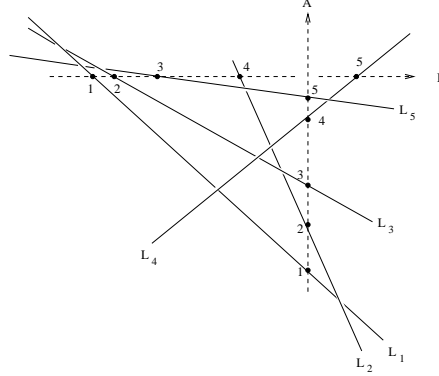


FIGURE 6. A spindle

A linking matrix X of a spindle C is easily computed as follows. Transform C isotopically into a spindle with oriented axis A and directrix B as above. Orient a line L_i from $L_i \cap A$ to $\sigma(L_i) = L_i \cap B$. A straightforward computation shows that the linking matrix X of this labeled and oriented configuration of skew lines has coefficients

$$x_{i,j} = \text{sign}((i - j)(\sigma(i) - \sigma(j)))$$

where $\text{sign}(0) = 0$ and $\text{sign}(x) = \frac{x}{|x|}$ for $x \neq 0$ and where σ is the corresponding spindle-permutation.

The linking matrix of Figure 6 is

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

Two spindle-permutations are *equivalent* (see [4, Section 15]), and give rise to isotopic spindle configurations, if they are equivalent under the equivalence relation generated by

(1) (Circular move)

$$\sigma \sim \mu \text{ if } \mu(i) = (s + \sigma((i + t) \pmod{n})) \pmod{n}$$

for some integers $0 \leq s, t < n$ (all integers are modulo n).

(2) (Vertical reflection of a block or local reversal) $\sigma \sim \mu$ if $\sigma([1, k]) = [1, k]$ and

$$\mu(i) = \begin{cases} k + 1 - \sigma(k + 1 - i) & i \leq k \\ \sigma(i) & i > k \end{cases}$$

for some integer $k \leq n$ (see Figure 7).

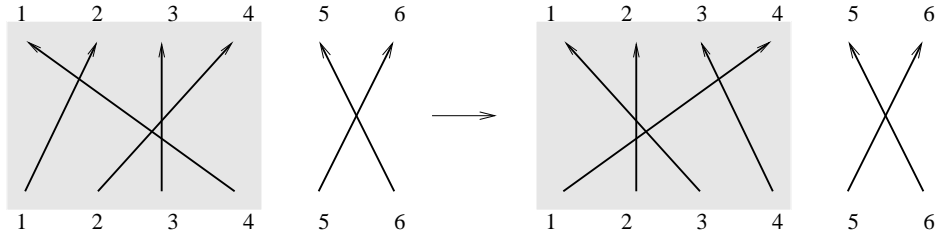


FIGURE 7. Vertical reflection of a block

(3) (Horizontal reflection of a block or local inversion) $\sigma \sim \mu$ if there exists an integer $1 < k \leq n$ such that $\sigma([1, k]) = [1, k]$ and

$$\mu(i) = \begin{cases} \sigma^{-1}(i) & i \leq k \\ \sigma(i) & i > k \end{cases}$$

(see Figure 8).

Permutations giving rise to linking matrices in a common switching class are spindle-equivalent by Corollary 1.2.

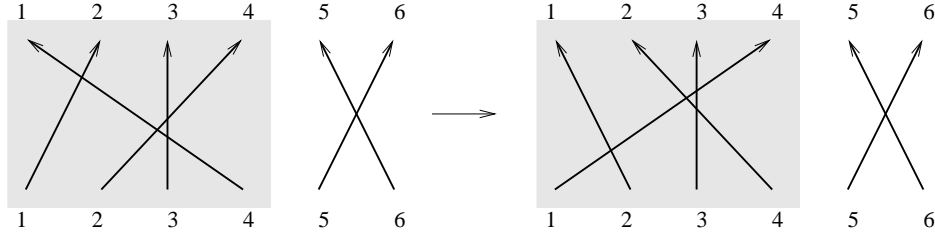


FIGURE 8. Horizontal reflection of a block

6. AN ALGORITHM FOR A SPINDLE-STRUCTURE

Given a switching class represented by a matrix X , the following algorithm constructs a spindle-permutation (which is unique up to spindle-equivalence) with linking matrix in the switching class of X or proves non-existence of such a permutation.

Algorithm 6.1.

Initial data. *A natural number n and a switching class represented by a symmetric matrix X of order n with rows and columns indexed by $\{1, \dots, n\}$ and coefficients $x_{i,j}$ satisfying*

$$\begin{aligned} x_{i,i} &= 0, & 1 \leq i \leq n, \\ x_{i,j} &= x_{j,i} \in \{\pm 1\}, & 1 \leq i \neq j \leq n. \end{aligned}$$

Initialization. *Conjugate the symmetric matrix X by the diagonal matrix with diagonal coefficients $(1, x_{1,2}, x_{1,3}, \dots, x_{1,n})$. Set $\gamma(1) = \gamma(2) = 1$, $\sigma(1) = 1$ and $k = 2$.*

Main loop. *Replace $\gamma(k)$ by $\gamma(k) + 1$ and set*

$$\sigma(k) = 1 + \#\{j \mid x_{\gamma(k),j} = -1\} + \sum_{s=1}^{k-1} x_{\gamma(s),\gamma(k)}.$$

Check the following conditions:

- (1) $\gamma(k) \neq \gamma(s)$ for $s \in \{1, \dots, k-1\}$.
- (2) $x_{\gamma(k),\gamma(s)} = \text{sign}(\sigma(k) - \sigma(s))$ for $s \in \{1, \dots, k-1\}$ (where $\text{sign}(0) = 0$ and $\text{sign}(x) = \frac{x}{|x|}$ for $x \neq 0$).
- (3) for $j \in \{1, \dots, n\} \setminus \{\gamma(1), \dots, \gamma(k)\}$ and for $s \in \{1, \dots, k-1\}$:
if $x_{j,\gamma(s)} x_{\gamma(s),\gamma(k)} = -1$, then $x_{j,\gamma(k)} = x_{j,\gamma(s)}$.

If all conditions are fulfilled, then:

if $k = n$, print all the data (mainly the spindle-permutation $i \mapsto \sigma(i)$ and perhaps also the conjugating permutation $i \mapsto \gamma(i)$) and stop.

if $k < n$, then set $\gamma(k+1) = 1$, replace k by $k+1$ and iterate the main loop.

If at least one of the above conditions is not fulfilled, then:

while $\gamma(k) = n$ replace k by $k-1$.

if $k = 1$, print “no spindle structure exists for this switching class” and stop.

if $k > 1$, iterate the main loop.

6.1. Explanation of the algorithm. The **initialization** is simply a normalization: we assume that the first row of the matrix represents the first line of a spindle-permutation σ normalized to $\sigma(1) = 1$ (up to a circular move, this can always be done for a spindle-permutation).

The **main loop** assumes that row number $\gamma(k)$ of X contains the linking numbers of the k -th line L_k (supposing a correct possible choice of the rows encoding the linking numbers of L_1, \dots, L_{k-1}). The image $\sigma(k)$ of k under a spindle-permutation is then uniquely defined and given by the formula used in the main loop.

One has to check three necessary conditions:

- The first condition checks that row number $\gamma(k)$ has not been used before.
- The second condition checks the consistency of the choice for $\gamma(k)$ with all previous choices.
- If the third condition is violated, the choice of rows $\gamma(1), \dots, \gamma(k)$ leads to a dead end. Indeed, we have then either $x_{\gamma(s), \gamma(k)} = 1, x_{j, \gamma(s)} = -1$ or $x_{\gamma(s), \gamma(k)} = -1, x_{j, \gamma(s)} = 1$ for some index $j \notin \{\gamma(1), \gamma(2), \dots, \gamma(k)\}$ and some natural integer $s < k$. In the first case, the line-segments L_s and L_k with $s < k$ representing σ graphically do not cross. This shows that any line-segment L_m with $m > k$ which crosses L_s has to cross L_k first. Since there must be at least one such line segment corresponding to the choice $\gamma(j) = m$, the algorithm must backtrack. The second case is similar.

The algorithm runs correctly even without checking out Condition (3). However, it loses much of its interest: An instance of Condition (3) (with fixed j, s, k) is violated with probability $\frac{1}{4}$ for a “random” choice (made e.g. by flipping a fair coin) of $x_{j, \gamma(s)}, x_{j, \gamma(k)} \in \{\pm 1\}$. This ensures fast running time in the average, as observed experimentally.

The algorithm, if successful, produces two permutations σ and γ . The linking matrix of the spindle permutation σ is in the switching class of X and γ yields a conjugation between these two matrices.

More precisely:

$$x_{\gamma(i),\gamma(j)} = \text{sign}((i - j)(\sigma(i) - \sigma(j)))$$

under the assumption $x_{1,i} = x_{i,1} = 1$ for $2 \leq i \leq n$.

Failure of the algorithm (indicated by the output “no spindle structure exists for this switching class”) proves non-existence of a spindle structure in the switching class of X .

ACKNOWLEDGMENTS

The second author wishes to thank the Institut Fourier where most of this work was done and Mikhail Zaidenberg for hosting his stay.

REFERENCES

- [1] R. Bacher and D. Garber, *Spindle configurations of skew lines*, Geom. Topo. **11** (2007), 1049–1081.
- [2] A. Borobia and V. F. Mazurovskii, *On diagrams of configurations of 7 skew lines of \mathbb{R}^3* , Amer. Math. Soc. Transl. (2) **173** (1996), 33–40.
- [3] A. Borobia and V. F. Mazurovskii, *Nonsingular configurations of 7 lines in \mathbb{RP}^3* , J. Knot Theory & Ramif. **6**(6) (1997), 751–783.
- [4] H. Crapo and R. Penne, *Chirality and the isotopy classification of skew lines in projective 3-space*, Adv. Math. **103** (1994), 1–106.
- [5] Y. V. Drobotukhina, *An analogue of the Jones polynomials for links in \mathbb{RP}^3 and a generalization of the Kauffman-Murasugi theorem*, Leningrad Math. J. **2**(3) (1991), 613–630.
- [6] S. I. Khashin, *Projective graph theory and configurations of lines*, J. Math. Sci. (New-York) **91**(6) (1998), 3532–3541.
- [7] S. I. Khashin and V. F. Mazurovskii, *Stable equivalence of real projective configurations*, Amer. Math. Soc. Transl. (2) **173** (1996), 119–140.
- [8] V. F. Mazurovskii, *Configurations of at most six lines in \mathbb{RP}^3* , Real Algebraic Geometry, Proceedings of the conference held in Rennes, France, June 24–28, 1991, Lecture Notes in Math. **1524** (1992), 354–371.
- [9] V. F. Mazurovskii, *Configurations of six skew lines* (Russian. English summary) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **167** (1988), Issled. Topol. 6, 121–134, 191; translation in: J. Soviet Math. **52**(1) (1990), 2825–2832.
- [10] J. J. Seidel, *Graphs and two-graphs*, Proc. 5th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Utilitas Mathematica Publishing Inc., Winnipeg, Canada, 1974.
- [11] N. J. A. Sloane (editor), *The On-Line Encyclopedia of Integer Sequences*, <http://www.research.att.com/~njas/sequences/>.
- [12] O. Y. Viro, *Topological problems concerning lines and points of three-dimensional space*, Soviet Math. Dokl. **2**(32) (1985), 528–531.
- [13] O. Y. Viro and Y. V. Drobotukhina, *Configurations of skew lines*, Leningrad Math. J. **1**(4) (1990), 1027–1050, or (updated version): <http://www.math.uu.se/~oleg/skewlines/index.html>, arXiv: math.GT/0611374.

- [14] T. Zaslavsky, *Glossary of Signed and Gain Graphs and Allied Areas*, The Electronic Journal of Combinatorics (1998), Dynamic Surveys in Combinatorics #DS9, 41 pages.

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